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FITTED DIFFEOMORPHISMS OF NON-SIMPLY CONNECTED MANIFOLDS

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TOPOLOGICAL ENTROPY roughly speaking measures the complexity of a dynamical system. In “Homology Theory and Dynamical Systems” [1] Shub and Sullivan defined a class of structurally stable diffeomorphisms called fitted which are C^0 dense in $\text{Diff}'(M)$. They show how to relate the dynamics of a fitted diffeomorphism to the induced chain map in homology theory, and using this determine the minimum topological entropy of fitted diffeomorphisms in each component of $\text{Diff}'(M)$, for simply connected manifolds of dimension greater than five. In this paper we extend these results to the non simply-connected case. The main tools are a sharpened statement of the Whitney cancelling lemma for handles of index 2 and $(n-2)$ using relative homotopy groups, and classical results of Whitehead concerning these groups [2]. Our methods also lead to an algebraic characterization of the chain complexes over $Z[\Pi_1]$ which arise from handle decompositions of high dimensional manifolds, extending a theorem of Smale’s in the simply connected case. It is a pleasure to acknowledge many helpful conversations with Michael Shub and Dennis Sullivan.

§0. BACKGROUND AND DEFINITIONS

First we recall the definition of fitted diffeomorphisms (see [1] for more details). Let M be a closed connected manifold, $\dim M = n$, and $\mathcal{H} = \{M_0 \subset M_1 \subset \cdots \subset M_n\}$ a handle decomposition of M . Recall this means $\overline{M_k} - \overline{M_{k-1}} \cong \bigcup_{i=1}^r (D_i^k \times D_i^{n-k})$, a disjoint union of k -handles ϕ_i^k with core discs $(D_i^k \times p)$ and transverse discs $(q \times D_i^{n-k})$. A diffeomorphism $f: M \rightarrow M$ is said to be transverse to \mathcal{H} ($f \in T_{\mathcal{H}}$) if (i) $f(M_k) \subset \text{int } M_k$ for all k (ii) $f(D_i^k \times 0) \cap (0 \times D_j^{n-k})$ for all k, i, j .

Let $\cap_{ij} = f(D_i^k \times 0) \cap (0 \times D_j^{n-k})$. For $f \in T_{\mathcal{H}}$ this will consist of a finite number of isolated points. Let $g_{ij}^k = \text{cardinality } (\cap_{ij})$ and record these *geometric intersection numbers* in a matrix $G_k = (g_{ij}^k)$. If we orient the core discs of \mathcal{H} we can also define *algebraic intersection numbers* a_{ij}^k by counting the intersection points with their signs $\text{ore}(p) = \pm 1$: $a_{ij}^k = \sum_{\cap_{ij}} \text{ore}(p)$. Clearly $g_{ij}^k \geq |a_{ij}^k|$. Let $C_*(M, \mathcal{H})$ be the chain complex induced by \mathcal{H} , $C_k = H_k(M_k, M_{k-1}; Z)$. For $f \in T_{\mathcal{H}}$, let $f_*: C_*(M, \mathcal{H}) \rightarrow C_*(M, \mathcal{H})$ be the induced chain map. The algebraic intersection matrices $A_k = (a_{ij}^k)$ represent f_* in the basis of oriented core discs.

Let f be transverse to \mathcal{H} . Shub and Sullivan construct a structurally stable diffeomorphism isotopic to f with the same intersection numbers. They say f is *fitted with respect to \mathcal{H}* if $f \in T_{\mathcal{H}}$, and: (i) the image of each core disc is a union of complete core discs, and the inverse image of each transverse disc is a union of complete transverse discs. (ii) f is uniformly expanding in the core directions, and uniformly contracting in the transverse directions (so f is hyperbolic on the invariant sets

$$K_i = \bigcap_{n \in Z} \overline{f^n(M_i - M_{i-1})}.$$

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THEOREM (Shub and Sullivan[1]). Suppose M, \mathcal{H} are as above, and $f \in \text{Diff}'(M)$, $r \geq 1$. (1) f is isotopic to a diffeomorphism fitted with respect to \mathcal{H} . (2) Fitted \Rightarrow Axiom A and Strong Transversality \Rightarrow Structurally Stable. (3) If f is fitted the entropy $h(f)$ is given by the formula

$$h(f) = \max_k \log s(G_k)$$

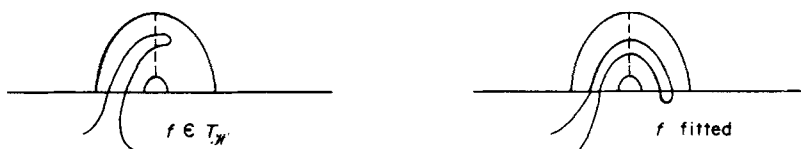
where $s(\cdot)$ is the spectral radius.

Note. The definition of fitted diffeomorphisms and (1) and (2) above extended earlier work of Smale[3] and of Shub and Williams.

The problem is to determine the minimum entropy of a fitted diffeomorphism in a component. Recall that if E and F are non-negative integral matrices, and $E_{ij} \geq F_{ij}$ for all i, j then $s(E) \geq s(F)$ [15]. Therefore the natural strategy is to reduce the g_{ij}^k by isotopy to minimize $h(f)$. This resolves into three steps. If an isotopy f_t , $0 \leq t \leq 1$, preserves the filtration of M —i.e. if $f_t(M_k) \subset \text{int } M_k$, for all t and k , then the a_{ij}^k are preserved and the best possible result would be to realize $g_{ij}^k = |a_{ij}^k|$. Next we characterize the (A_k) which can occur in the component of f . Shub and Sullivan show that an endomorphism $E: C_*(M, \mathcal{H}) \rightarrow C_*(M, \mathcal{H})$ is realized as the algebraic intersection matrices of $g \in T_{\mathcal{H}}$ isotopic to f if and only if E is chain homotopic to f_* . Finally we characterize the chain complexes which arise from handle decompositions of M . When $\Pi_1(M) = 0$ and $\dim M \geq 6$ Shub and Sullivan show these steps reduce the minimum fitted entropy problem to a non-trivial but purely algebraic problem.

In this paper we carry out the same program when $\Pi_1(M) \neq 0$, however the algebra required is considerably more complicated. The appropriate algebraic models are dictated by the Whitney cancelling lemma. In the middle dimensions $3 \leq k \leq n-3$ we can use the chain groups on the universal cover $C_*(\tilde{M}, \mathcal{H})$; otherwise we use homotopy groups $\Pi_2(M_2, M_1, x_0)$ and $\Pi_1(M_1, x_0)$.

If C_* is a free $Z[\Pi_1]$ complex we will say C_* admits homotopy groups if the bottom of the complex, $C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ arises from a free presentation of Π_1 . We obtain the following characterization of the chain complexes of M .



THEOREM. $\dim M \geq 6$, \mathcal{H} a handle decomposition of M with one 0-handle and one n -handle, and C_* a free $Z[\Pi_1]$ complex such that C_* and its dual C^* admit homotopy groups. Then C_* is realized by a handle decomposition of M if and only if there exists a chain homotopy equivalence $G: C_*(\tilde{M}, \mathcal{H}) \rightarrow C_*$ with Whitehead torsion $\tau(G) = 0$. In that case there also exists a diffeomorphism g isotopic to Id_M such that $g_* = G^\dagger$. \square

Let C_*, G be as above; we will say a chain map $E_*: C_* \rightarrow C_*$ is an endomorphism of f if $Gf_* = E_*G$. The minimum fitted entropy problem reduces to the following algebraic problem.

THEOREM. Let M, \mathcal{H} be as above and $f \in T_{\mathcal{H}}$. The infimum of the entropy of fitted

[†]We need to require that G_n satisfy a mild condition satisfied by induced chain maps of cellular homeomorphisms.

diffeomorphisms in the component of f , with a single source and a single sink, is given by

$$\text{infimum} = \inf_{E_* \text{ an endomorphism of } f} \log s(|E_*|)$$

where the spectral radius is evaluated on C_* for $3 \leq k \leq n-3$, and on relative homotopy groups otherwise.

§1. ALGEBRAIC AND GEOMETRIC INTERSECTION NUMBERS

When $\Pi_1(M) \neq 0$ it is natural to define algebraic intersection numbers on the chain complex of the universal cover $C_*(\tilde{M}, \tilde{\mathcal{H}})$. In the middle dimensions $3 \leq k \leq n-3$ the absolute value of these algebraic intersection numbers can be realized geometrically, using the Whitney lemma as in the proof of the s -cobordism theorem, but in the extreme dimensions this runs into difficulties. The correct models in dimensions 1, 2 are the homotopy groups $\Pi_1(M_1, x_0)$ and $\Pi_2(M_2, M_1, x_0)$. The top dimensions $n-1$ and $n-2$ can then be handled by duality.

Choose a base point x_0 in $\text{int } M_0$, and for each handle a base path Γ_i^k from x_0 to ϕ_i^k . Let $\Pi_1 = \Pi_1(M)$, and let $Z[\Pi_1]$ be the integral group ring. With the chosen base paths and orientations, the core discs $(D_i^k \times 0)$ of \mathcal{H} are a basis for $C_*(\tilde{M}, \tilde{\mathcal{H}})$ as a complex over $Z[\Pi_1]$. To each point $p \in \cap_{ij} = f(D_i^k \times 0) \cap (0 \times D_j^{n-k})$, we associate an element $\alpha(p) \in \Pi_1(M, x_0)$ as follows. Let $\Gamma_i^k|_p$ be Γ_i^k extended to p by some path in ϕ_i^k . Then $\alpha(p) = [f(\Gamma_i^k|_{f^{-1}(p)})(\Gamma_j^k|_p)^{-1}]$. We define algebraic intersection numbers in $Z[\Pi_1]$ by $\tilde{a}_{ij}^k = \sum_{p \in \cap_{ij}} \text{ore}(p)\alpha(p)$. For $f \in T_{\mathcal{H}}$ the algebraic intersection matrices $\tilde{A}_k = (\tilde{a}_{ij}^k)$ represent $f_*: C_*(\tilde{M}, \tilde{H}) \rightarrow$ in the chosen basis. We define an “absolute value” on $Z[\Pi_1]$ by

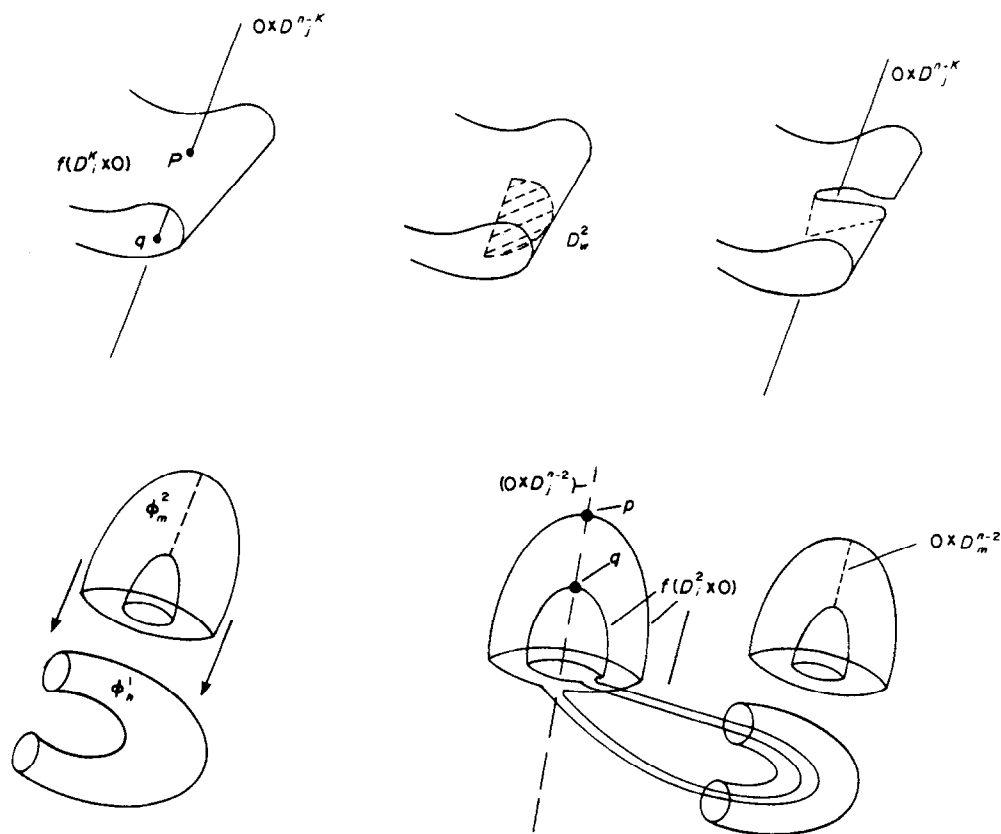
$$\left| \sum_{\alpha \in \Pi_1} \epsilon_\alpha \alpha \right| = \sum_{\alpha \in \Pi_1} |\epsilon_\alpha|.$$

LEMMA 1 (Shub and Sullivan[1]). *Suppose $\dim M \geq 5$, $3 \leq k \leq n-3$, and $f \in T_{\mathcal{H}}$. Then f is isotopic to $f' \in T_{\mathcal{H}}$ such that $g_{ij}^k(f') = |\tilde{a}_{ij}^k(f')|$ for all i, j , and the other geometric intersection numbers are unchanged.*

Proof. This is a straightforward application of the Whitney lemma[4]. The contribution to \tilde{a}_{ij}^k of a pair $p, q \in \cap_{ij}$ cancels in $Z[\Pi_1]$ if and only if $\alpha(p) = \alpha(q)$ and $\text{ore}(p) = -\text{ore}(q)$. If so, form a loop $\ell(p, q)$ consisting of an embedded arc from p to q in $f(D_i^k \times 0)$ followed by an embedded arc from q to p in $(0 \times D_j^{n-k})$. Since $\alpha(p) = \alpha(q)$ in $\Pi_1(M) \cong \Pi_1(M_k)$, $\ell(p, q)$ is spanned by a possibly singular 2-disc D_w^2 in M_k . Since $\dim M \geq 5$ we can assume D_w^2 is embedded. Slide the arc from p to q in $f(D_i^k \times 0)$ back across D_w^2 to eliminate p and q . This extends to an isotopy of f which is fixed except in a small neighborhood of $f^{-1}(D_w^2)$.

By general position we can prevent extraneous intersections of D_w^2 with transverse discs of k -handles ($k \geq 3$), or with images of k -core discs ($k \leq n-3$). Therefore no new intersections are created during the isotopy and the lemma follows by a finite induction. \square

When $k = 2$ the last step in this proof breaks down. D_w^2 can meet transverse discs of 2-handles in general position, in which case new points of intersection would be created in pairs by the Whitney isotopy. Suppose ϕ_h^1 and ϕ_m^2 are handles such that the boundary of ϕ_m^2 kills the class represented by ϕ_h^1 in $\Pi_1(M, x_0)$, and $p, q \in \cap_{ij}$ cancel



in \tilde{a}_{ij}^2 . It can happen that $\ell(p, q)$ contracts in M_2 , but only across the transverse disc of ϕ_m^2 .

The $Z[\Pi_1]$ intersection numbers do not detect that p and q differ by the action of ϕ_n^1 . The cancelling which can be achieved without creating new intersections is measured by the relative homotopy group $\Pi_2(M_2, M_1, x_0)$, which admits the action of the free group $\Pi_1(M_1, x_0)$.

The relative homotopy groups $\Pi_i(M_i, M_{i-1}, x_0)$ were studied by Whitehead in the case of a CW complex, and we will follow his notation[2]. We will write ρ_i for $\Pi_i(M_i, M_{i-1}, x_0)$ and $d_i: \rho_i \rightarrow \rho_{i-1}$ for the boundary. Of course, $d_k d_{k+1} = 0$, $k \geq 3$, and $d_2 d_3 = 1$. There are natural projections $h_i: \rho_i \rightarrow C_i$ which are essentially the Hurewicz homomorphisms and are isomorphisms for $i \geq 3$. $\rho_1 = \Pi_1(M_1, x_0)$ is a free group and acts on ρ_2 . We will write $\rho(M, \mathcal{H})$ for the complex (ρ_i, d_i) . Essentially we are adding a free presentation of Π_1 to $C_*(\tilde{M}, \tilde{\mathcal{H}})$

$$\begin{array}{ccccc}
 & & d_2 & & \\
 & & \rho_2 \longrightarrow & \rho_1 & \\
 & \downarrow & & \downarrow & \\
 \rightarrow C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 \xrightarrow{\partial_1} C_0.
 \end{array}$$

It is easier to deal with dimensions 1, $n-1$ if we assume \mathcal{H} has a single 0-handle and a single n -handle. As a consequence, the minimum entropy we detect will be the minimum for fitted diffeomorphisms with a single source and a single sink. We choose a base point x'_0 for the dual decomposition \mathcal{H}' in the interior of the n -handle, and require that for $f \in T_{\mathcal{H}}$, $f(x_0) = x_0$ and $f(x'_0) = x'_0$. For $f \in T_{\mathcal{H}}$, $f_{\#}: \rho(M, \mathcal{H}) \rightarrow \rho(M, \mathcal{H})$ will be the induced homomorphism.

Following Whitehead we will write ρ_2 additively, although it is not in general abelian, and write the action of ρ_1 on ρ_2 multiplicatively. For $x \in \rho_1$, $A \in \rho_2$ we have $d_2(xA) = x d_2(A) x^{-1}$, so $d_2(\rho_2)$ is a normal subgroup of ρ_1 . We will write $\bar{\rho}_1$ for the quotient $\rho_1/d_2(\rho_2) \cong \Pi_1$ and \bar{x} for the coset of $x \in \rho_1$. Let $[\phi_i^k] \in \rho_k$ be the element represented by the core disc, appropriately equipped with base path and orientation. ρ_2 is generated by all elements of the form $\pm x[\phi_i^2]$, $x \in \rho_1$. The relations are of the form

$$A + B - A = d_2(A)B \quad (\star)$$

$A, B \in \rho_2$. Whitehead called a group of this type a *free crossed module*.

Because of the relations (\star) there is no canonical way to define the "absolute value" of algebraic intersections recorded by $f_{\#2}$. Given a word W representing $f_{\#2}[\phi_i^2]$,

$$W = \epsilon_1 x_1 [\phi_{k_1}^2] + \cdots + \epsilon_r x_r [\phi_{k_r}^2] \quad (\star\star)$$

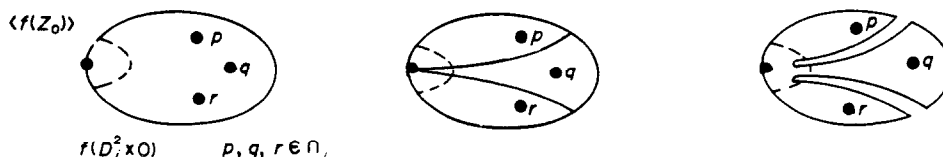
$\epsilon_k = \pm 1$, $x_k \in \rho_1$, we let

$$|W_{ij}| = \sum_{k_s=j} |\epsilon_s|$$

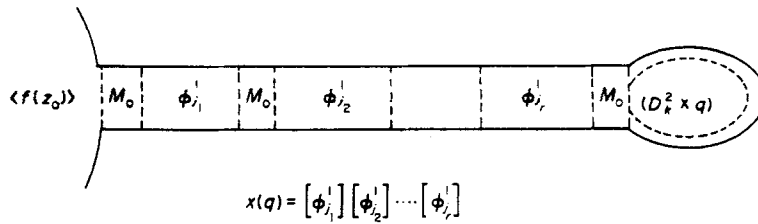
that is, we count occurrences of $[\phi_i^2]$ in W .

LEMMA 2. Suppose $\dim M \geq 5$ and $f \in T_{\mathcal{K}}$. If W is a word representing $f_{\#2}[\phi_i^2]$ then f is isotopic to $f' \in T_{\mathcal{K}}$ such that, for all j , $g_{ij}^2(f') = |W_{ij}|$ and the other geometric intersection numbers are unchanged.

Proof. We first isotope $f|(D_i^2 \times 0)$ so it is unambiguously associated with a word of the form $(\star\star)$. Let $\cap_i = \bigcup_k \cap_{ik}$. By [1, Theorem 1.1] we can assume f is already fitted, so $f(D_i^2 \times 0) \cap \overline{M_2 - M_1}$ is a union of complete core discs, one for each point $q \in \cap_i$. By abuse of notation, if $q \in \cap_{ik}$ we will call this core disc $(D_k^2 \times q)$. By [1, Proposition 1.2] we can assume the 2-handles are attached in a good way, so $(S_i^1 \times 0) \cap \partial M_0 \neq \emptyset$, and choose a base point for $(D_i^2 \times 0)$, $z_0 \in (S_i^1 \times 0) \cap \partial M_0$. Let $\langle f(z_0) \rangle$ be the component of $f(D_i^2 \times 0) \cap M_0$ containing $f(z_0)$. Choose arcs from $f(S_i^1 \times 0)$ to $f(z_0)$ in $f(D_i^2 \times 0) \cap M_1$ which separate the points of \cap_i and meet only at $f(z_0)$. Then "unzip" $f(S_i^1 \times 0)$ along these arcs by an isotopy, so each disc $(D_k^2 \times q)$ lies in a distinct component of $(f(D_i^2 \times 0) - \langle f(z_0) \rangle)$.



Continuing this process we can ensure that each component of $f(D_i^2 \times 0) \cap \overline{M_1 - M_0}$ meets $f(S_i^1 \times 0)$. If we make f fitted on $(D_i^2 \times 0)$, each of the discs $(D_k^2 \times q)$ will be connected to $\langle f(z_0) \rangle$ in $f(D_i^2 \times 0)$ through a sequence of intersections of $f(D_i^2 \times 0)$ with 1-handles. Let $x(q) \in \rho_1$ be the class represented by this sequence of 1-handles.



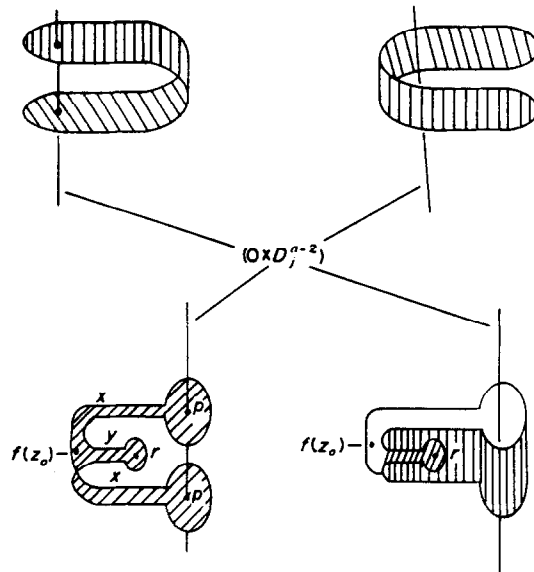
If we take a little care with base points we obtain a word

$$W' = \epsilon_1 x_1 [\phi_{k_1}^2] + \cdots + \epsilon_r x_r [\phi_{k_r}^2]$$

representing $f_{\#}[\phi_i^2]$, and a 1-1 correspondence $g: \cap_i \rightarrow \{1, \dots, r\}$ such that $x_{g(i)} = x(q)$. In this case we will say $f|(D_i^2 \times 0)$ is in normal form associated with the word W' .

Suppose then $f|(D_i^2 \times 0)$ is in normal form associated with W' and $W' \Rightarrow W$ by one application of the relations (\star) . We show f is isotopic to f' in normal form associated with W . For simplicity, suppose $W' = x[\phi_j^2] + y[\phi_k^2] - x[\phi_j^2]$ and $W = xd_2[\phi_j^2]x^{-1}y[\phi_k^2]$, $x, y \in \rho_1$. Let $p, q \in \cap_{ij}$ correspond to the terms $\pm[\phi_j^2]$. Since $xx^{-1} = 1$ in $\Pi_1(M_1, x_0)$ $\ell(p, q)$ is spanned by a 2-disc Δ in $M_1 \cup \phi_j^2$. Therefore extraneous intersections of Δ with transverse discs of 2-handles can be avoided. In this context the Whitney isotopy becomes "two ways to span a folded circle with a 2-disc".

Let $r \in \cap_{ik}$ correspond to the term $y[\phi_k^2]$ of W' . Notice that an arc from $f(z_0^i)$ to r representing y must cross $\ell(p, q)$. After the isotopy r will be associated with $xd_2([\phi_j^2])x^{-1}y$ as in the following figure.



The isotopy we construct of $f|(D_i^2 \times 0)$ can be arranged to avoid the images of other core discs of dim 0, 1, 2. By the Extension of Isotopy Theorem [4] we obtain an isotopy defined on all of M which leaves all the other geometric intersection numbers unchanged. \square

This leaves the cases $k = 1, (n - 1)$. Since we assume \mathcal{H} has a single 0-handle, ρ_1 is the free group on the 1-handles. Given a word W in 1-handles representing $f_{\#}[\phi_i^1]$

$$W = [\phi_{k_1}^1]^{\epsilon_1} \cdots [\phi_{k_r}^1]^{\epsilon_r}$$

let $|W_{ij}| = \sum_{k_j=j} |\epsilon_s|$. Provided $\dim M \geq 4$ it is easy to see that $f \in T_n$ can be isotoped to realize the geometric intersections recorded by the absolute value of the reduced words representing $f_{\#}[\phi_i^1]$.

§2. CHAIN HOMOTOPY

Next we determine the chain maps $E: \rho(M, \mathcal{H}) \rightarrow \rho(M, \mathcal{H})$ realized by $g \in T_n$ isotopic to f . A necessary and sufficient condition is that E be chain homotopic to $f_{\#}$, provided E_n satisfies an algebraic condition corresponding to injectivity of g . The proof is the same as [1, Prop. 2.1] but the algebra is more involved.

The appropriate definition of chain homotopy was formulated by Whitehead. We recall the definitions; for more details see [2]. A homotopy system $\rho = (\rho_i, d_i)$, $i \geq 1$, is a (non-abelian) chain complex, where ρ_1 is a free group, ρ_2 is a free crossed module, and for $k \geq 3$, ρ_k is a free $Z[\bar{\rho}_1]$ module. We assume each ρ_i is given with a selected finite basis. We allow ρ_1 to act on ρ_k , $k \geq 3$, by $xA = \bar{x}A$, $x \in \rho_1$, $A \in \rho_k$.

Let G, G' be additive groups which admit, respectively, the actions of multiplicative groups γ, γ' and let $E: G \rightarrow G'$, $F: \gamma \rightarrow \gamma'$ be homomorphisms. E is an operator homomorphism associated with F if for $x \in \gamma$, $A \in G$, $E(xA) = F(x)E(A)$. A function $k: \gamma \rightarrow G'$ is a crossed homomorphism associated with F if for $x, y \in \gamma$, $k(xy) = k(x) + F(x)k(y)$. Let ρ, ρ' be homotopy systems. A homomorphism $E: \rho \rightarrow \rho'$ is a chain map such that E_k , $k \geq 2$, is an operator homomorphism associated with E_1 . Two homomorphisms E, E' are chain homotopic if there exist $\omega \in \rho'_1$ and a family of maps $\xi_k: \rho_{k-1} \rightarrow \rho'_k$, $k \geq 2$, where ξ_2 is a crossed homomorphism, and ξ_k , $k > 2$, an operator homomorphism, all associated with E_1 , and

$$\begin{aligned} \omega E'_k - E_k &= d'_{k+1} \xi_{k+1} + \xi_k d_k, \quad k \geq 2 \\ (\omega E'_1(x) \omega^{-1}) E_1(x)^{-1} &= d'_2 \xi_2(x) \quad \text{for } x \in \rho_1. \end{aligned}$$

In that case we will write $E \underset{\xi, \omega}{\simeq} E'$.

With a homotopy system ρ we associate a chain complex $C_*(\rho) = (C_i, \partial_i)$, $i \geq 0$. For $i \geq 1$, C_i is a free $Z[\bar{\rho}_1]$ module with a basis in 1-1 correspondence with the chosen basis of ρ_i , and C_0 has a single basis element a_0 . Define projections $h_i: \rho_i \rightarrow C_i$ to be maps of the appropriate algebraic type induced by the correspondence of bases. For $k \geq 3$, h_k is a $Z[\bar{\rho}_1]$ module isomorphism, h_2 is abelianization, and h_1 is a crossed homomorphism associated with the quotient $\rho_1 \rightarrow \bar{\rho}_1$, i.e. $h_1(xy) = h_1(x) + \bar{x}h_1(y)$. A chain map $F: C_*(\rho) \rightarrow C_*(\rho')$ will be a family of operator homomorphisms associated with a homomorphism $\theta: \bar{\rho}_1 \rightarrow \bar{\rho}'_1$ such that $\partial_k F_k = F_{k-1} \partial_k$ and $F(a_0) = a'_0$. A homomorphism $E: \rho \rightarrow \rho'$ induces a chain map $E_*: C_*(\rho) \rightarrow C_*(\rho')$, conversely E will be called a lift of E_* . Chain homotopy is defined on $C_*(\rho)$ in the usual way. A chain homotopy (ξ, ω) from ρ to ρ' induces one downstairs $(\xi_*, \bar{\omega})$ by $\xi_* h = h' \xi$ and $\xi_{*1}(a_0) = h'_1(\omega)$.

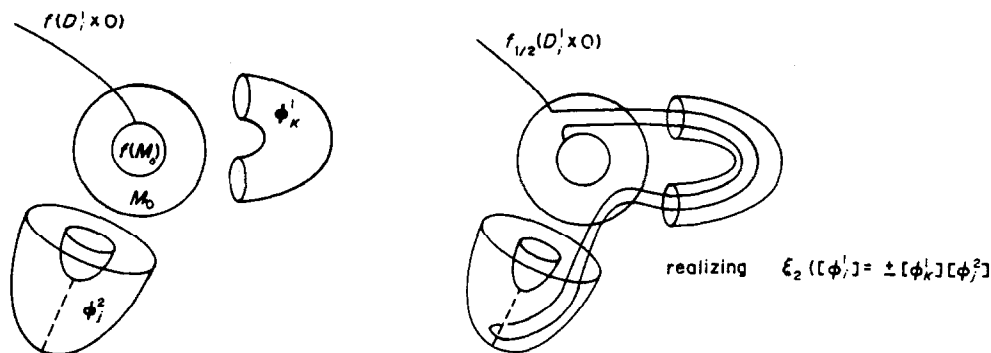
THEOREM ([2], Theorems 9, 10). *A chain map $F: C_*(\rho) \rightarrow C_*(\rho')$ has at least one lift; a lift is uniquely determined by its values on ρ_1 . Two lifts of the same chain map are chain homotopic upstairs. If $E, E': \rho \rightarrow \rho'$ are homomorphisms, then $E \underset{(\xi, \omega)}{\simeq} E'$ if*

and only if $E_ \underset{(\xi_*, \bar{\omega})}{\simeq} E'_*$ downstairs.*

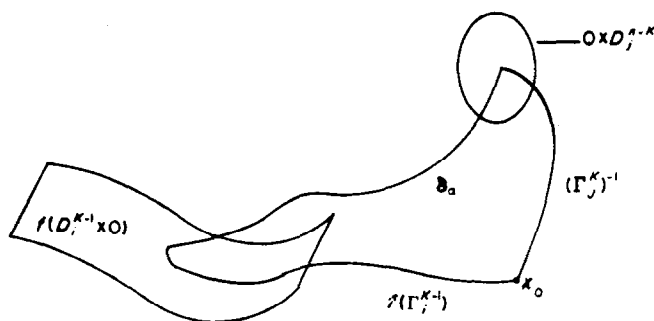
Definition. Let ρ, ρ' be homotopy systems of dimension n , such that ρ_n, ρ'_n each have a single basis element $a_1^n, (a_1^n)'$. $E: \rho \rightarrow \rho'$ is *monic* if $E_n(a_1^n) = \pm \alpha (a_1^n)'$, $\alpha \in \rho'_1$.

LEMMA 3. Suppose $\dim M \geq 4$, and \mathcal{H} is a handle decomposition with one 0-handle and one n -handle. Let $f \in T_{\mathcal{H}}$ and $E: \rho(M, \mathcal{H}) \rightarrow \rho(M, \mathcal{H})$. f is isotopic to a diffeomorphism $g \in T_{\mathcal{H}}$ such that $g_{\#} = E$ if and only if E is monic and $f_{\#} \approx E$.

Proof. Suppose $f_{\#} \approx E$ by (ξ, ω) . To realize $\omega \in \rho_1$ we pull the image $f_t(x_0)$, $0 \leq t \leq 1$, through the sequence of 1-handles representing ω . It is easy to check this has the desired effect on $f_{\#}$. For $k \geq 2$ we show how to create the required linking numbers to realize ξ_k one by one, and the result follows by finite induction. Suppose $\xi_2([\phi_i^1]) = \epsilon \alpha [\phi_j^2]$, $\epsilon = \pm 1$, $\alpha \in \rho_1$. Take hold of an arc in $f(D_i^1 \times 0) \cap M_0$ and pull a loop through the word α in 1-handles, then along the base path of ϕ_j^2 and up into ϕ_j^2 , and create local linking number ϵ with $(0 \times D_j^{n-2})$. Then push $f(D_i^1 \times 0)$ back into $\text{int } M_1$ by radial isotopy in ϕ_j^2 .



Suppose $3 \leq k \leq n-1$ and $\xi_k([\phi_i^{k-1}]) = \epsilon \alpha [\phi_j^k]$ where $\alpha \in \Pi_1(M)$. Choose an arc δ_α in $M_{k-1} \cup \phi_j^k$ from $f(D_i^{k-1} \times 0)$ to $(0 \times D_j^{n-k})$ such that $[f(\Gamma_i^{k-1})\delta_\alpha(\Gamma_j^k)^{-1}] = \alpha$. Provided $k \leq n-1$ by general position δ_α will miss the images of other $(k-1)$ core discs. Pull a "tube" in $f_t(D_i^{k-1} \times 0)$ up along δ_α into ϕ_j^k and create local linking number ϵ with $(0 \times D_j^{n-k})$. Again it is easy to verify that, after pushing $f_t(D_i^{k-1} \times 0)$ back down into $\text{int } M_{k-1}$, $(f_1)_{\#} - (f_0)_{\#} = d'_{j+1}\xi_{j+1} + \xi_j d_j$, $j = K, K-1$ [5, 2.7].



When $k = n$ the appeal to general position fails. If $\xi_n([\phi_i^{n-1}]) = \epsilon \beta [\phi_j^n]$ and δ_β is the chosen arc, $\text{int } \delta_\beta$ can meet $f(D_i^{n-1} \times 0)$ in general position. If we used such an arc then $f_t(D_i^{n-1} \times 0)$ would also be linked with $(0 \times D_i^0)$. We avoid this problem by using the homotopy system of the dual decomposition $\mathcal{H}' = \{M'_0 \subset M'_1 \subset \dots \subset M'_n\}$. Any ξ_n that can be realized by isotopy of f must correspond to a word $\omega' \in \Pi_1(M'_1, x'_0)$. If $x'_0 = (0 \times D_i^0)$ when $f_t(D_i^{n-1} \times 0)$ is pulled across $(0 \times D_i^0)$ the inverse image $f_t^{-1}(x'_0)$ is pulled through the dual 1-handle $(\phi_i^1)'$.

Let $C^*(\rho)$ be the dual chain complex $C^k = \text{Hom}_{Z[\Pi_1]}(C_k, Z[\Pi_1])$. Up to a factor

$(-1)^k$ in ∂^{n-k+1} , and an identification of $\Pi_1(M, x_0)$ with $\Pi_1(M, x'_0)$. $C^*(\rho(M, \mathcal{H})) = C^*(\tilde{M}, \tilde{\mathcal{H}})$ is isomorphic to $C_*(\rho(M, \mathcal{H})) = C_*(\tilde{M}, \tilde{\mathcal{H}})$ [6, 7]. Likewise for $f \in T_{\mathcal{H}}$ $(f^{-1})_*: C_*(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C_*(\tilde{M}, \tilde{\mathcal{H}})$ is identified with $f^*: C^*(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C^*(\tilde{M}, \tilde{\mathcal{H}})$. If $E: \rho(M, \mathcal{H}) \rightarrow \rho(M, \mathcal{H})$ is monic then $E^*: C^*(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C^*(\tilde{M}, \tilde{\mathcal{H}})$ is a chain map in our sense, i.e. $E^*(a'_0) = a'_0$, and we can apply the theorem of Whitehead quoted above. It is easy to show that if $E, F: C_*(\rho) \rightarrow C_*(\rho)$ and $E = F$ by $(\xi_*, \bar{\omega})$ then $E^* = F^*$ by $(\xi^*, (\bar{\omega})^{-1})$ [5].

Suppose we have realized ξ_k , $k \leq n-1$ and $f_{\#} \approx E$ by $(\xi_n, 1)$, $1 \in \rho_1$. Since $\dim M \geq 4$ it is equivalent to realize $(\xi_n)_*$ on $C_*(\rho)$. But $f_{\#} \approx E^*$ by $(\xi_n)^*: C^0 \rightarrow C^1$ and by the theorem of Whitehead we can lift $(\xi_n)^*$ to a chain homotopy defined on $\rho(M, \mathcal{H})$. A lift of $(\xi_n)^*$ is an element $\omega' \in \rho'_1$ and this is the required word in dual 1-handles. We realize ω' by an isotopy of f^{-1} , f_t^{-1} , $0 \leq t \leq 1$. Therefore $(f_t^{-1})_* = E^*$ and $(f_t)_* = E_*$. It follows from our construction that $f_t^{-1}(M'_i) \subset \text{int } M'_i$ for $i \geq 1$, $0 \leq t \leq 1$, so $f_t(M_{n-i-1}) \subset \text{int } M_{n-i-1}$. Since $n \geq 4$, $f_t(M_i) \subset \text{int } M_i$ for $i = 1, 2$ and $f_t(x_0) = x_0$, so $f_{\#1}$ and $f_{\#2}$ are unchanged. Therefore $(f_1)_{\#} = E$.

Conversely, if g is a diffeomorphism transverse to \mathcal{H} , $g_{\#}$ is clearly monic and, if f is isotopic to g , then $f_{\#}$ is chain homotopic to $g_{\#}$ [2, Theorem 5]. \square

Definition. Let $E: \rho \rightarrow \rho$ be a homomorphism. We define the spectral radius $s(|E|)$. For $k \geq 3$, E_k is associated with a matrix (E_{ij}^k) over $Z[\bar{\rho}_1]$. Let $s(|E_k|) = s(|(E_{ij}^k)|)$. When $k = 1$, let $(|W_{ij}^1|)$ be the matrix obtained from the reduced words W_i^1 representing $E_1(a_i^1)$ and let $s(|E_1|) = s(|(W_{ij}^1)|)$. When $k = 2$, given a collection of words $\{W_i^2\}$ representing $E_2(a_i^2)$ associate a non-negative integral matrix $(|W_{ij}^2|)$. If $\{\bar{W}_i^2\}$ is another collection such that $s(|(\bar{W}_{ij}^2)|) < s(|(W_{ij}^2)|)$ then $|\bar{W}_{ij}^2| < |W_{ij}^2|$, for some i, j . It is not hard to show that only a finite chain of such matrices $(|W_{ij}^2|)$ can exist, so we can define $s(|E_2|)$ to be the minimum of $s(|(W_{ij}^2)|)$ over all such collections of words. Finally, let $s(|E|) = \max_k s(|E_k|)$.

Combining the previous lemmas we obtain a structure theorem for diffeomorphisms transverse to \mathcal{H} . Let ρ, ρ' be homotopy systems such that $C^*(\rho) \approx C^*(\rho')$. By an endomorphism (E, E') of (ρ, ρ') we mean a pair $E: \rho \rightarrow \rho$, $E': \rho' \rightarrow \rho'$ such that $(E^*) = (E')^*$.

THEOREM 1. Suppose $\dim M \geq 5$, and \mathcal{H} is a handle decomposition of M with one 0-handle and one n -handle. Let $f \in T_{\mathcal{H}}$. (a) An endomorphism (E, E') of $(\rho(M, \mathcal{H}), \rho(M, \mathcal{H}))$ is realized by a diffeomorphism $g \in T_{\mathcal{H}}$ isotopic to f if and only if $f_{\#} \approx E_{\#}$. (b) A collection of integral matrices $\{G_0, \dots, G_n\}$, $G_k = (g_{ij}^k)$ is realized as the geometric intersection matrices of such a diffeomorphism g if and only if there exists an endomorphism (E, E') as in part A and a collection of words W_i^k representing E_k , $k = 1, 2$, and E'_{n-k} , $k = n-1, n-2$, such that

$$\begin{aligned} g_{ij}^k &= |E_{ij}^k| + 2m, \quad m \in \mathbb{Z}^+, \quad 3 \leq k \leq n-3 \\ &= |W_{ij}^k| + 2m, \quad m \in \mathbb{Z}^+, \quad k = 1, 2, n-1, n-2 \end{aligned}$$

and $g_{11}^0 = g_{11}^n = 1$. (c) If (E, E') is as in Part A then f is isotopic to a diffeomorphism g fitted with respect to \mathcal{H} such that

$$h(g) = \max \{ \log s(|E|), \log s(|E'|) \}.$$

Proof. (a) If $f_{\#} \approx E_{\#}$ by the theorem of Whitehead $f_{\#} \approx E$, while E is monic since $(E)^* = (E')^*$. By Lemma 3, f is isotopic to $h \in T_{\mathcal{H}}$ such that $h_{\#} = E$. Therefore

$(h^{-1})_* = (E')_*$ and $(h^{-1})_* = E'$, by a chain homotopy of the form (ω', ξ') . By Lemma 3 again, h is isotopic to g such that $(g^{-1})_* = E'$, and since $\dim M \geq 5$ realizing (ω', ξ') leaves h_* unchanged on $\rho_1(M, \mathcal{H})$ and $\rho_2(M, \mathcal{H})$. Therefore $g_* = E$ as well. The converse is immediate as before. (b) Given the endomorphism (E, E') and the words W_i^k , by Part (a) and Lemmas 1 and 2 f is isotopic to a diffeomorphism $h \in T_\pi$ such that $g_{ij}^k(h) = |E_{ij}^k|$, $3 \leq k \leq n-3$, and $g_{ij}^k(h) = |W_{ij}^k|$, $k = 1, 2, n-1, n-2$. By introducing cancelling loops in $f(D_i^k \times 0)$, given g_{ij}^k we can realize $g_{ij}^k + 2$. Conversely, given $g \in T_\pi$ isotopic to f , g_* , $(g^{-1})_*$ is the required endomorphism, since the $|E_{ij}^k|$, $|W_{ij}^k|$ arise from the geometric intersection numbers g_{ij}^k by cancelling which occurs in pairs. (c) This follows from (b) and the result that, for f fitted, $h(f) = \max_k \log s(G_k(f))$.

§3. SIMPLE HOMOTOPY THEORY

To complete the reduction to algebra we need to characterize the chain complexes which arise from handle decomposition of M . In the simply connected case this question was essentially answered by Smale as a consequence of the h -cobordism theorem.

THEOREM (Smale[8]). *Let M be simply connected and $\dim M \geq 6$. There exists a handle decomposition of M with the minimum number of handles of each dimension consistent with the homology.* \square

Shub and Sullivan pointed out that Smale's argument actually proves a more general result: let M be as above and C_* a free finitely generated chain complex over Z such that $\dim C_* = \dim M$, $H_*(C_*) \cong H_*(M; Z)$ and $C_1 = C_{n-1} = 0$. Then C_* is realized by a handle decomposition of M [1, 9].

In the non-simply connected case it is natural to work in the universal cover \tilde{M} and consider chain complexes over the group ring $Z[\Pi_1]$. We start with the chain complex of a given handle decomposition \mathcal{H} and modify it by operations on the handles, as in the proof of the s -cobordism theorem. Since our statement of the Whitney lemma works all the way down we do not need to eliminate 1- or $(n-1)$ -handles. This leads to a characterization based on simple homotopy theory.

In [10, §17] Whitehead defined simple homotopy theory for homotopy systems. In this section we will recall the definitions and prove some preliminary algebraic lemmas.

Henceforth we will assume a homotopy system ρ is given with a class of preferred bases for each ρ_k , $\{a_i^k, \dots, a_{r_k}^k\}$ determined up to order, sign, and for $k \geq 2$, multiplication by elements of ρ_1 . By a *based $Z[\Pi_1]$ complex* we will mean a free, finitely generated chain complex over $Z[\Pi_1]$, $0 \rightarrow C_n \rightarrow \dots \rightarrow C_0 \rightarrow 0$, where each C_i is given with a class of preferred bases, determined up to order, sign, and for $i \geq 1$ multiplication by elements of Π_1 . If ρ is a homotopy system, $C_*(\rho)$ is a based $Z[\rho_1]$ complex in the obvious way.

A homomorphism $F: \rho \rightarrow \rho'$ is a *simple isomorphism* ($F: \rho \approx \rho'$) if the image of a preferred basis is a preferred basis. Let a_i^r, a_j^{r-1} be basis elements, $r \geq 3$, such that $d_r(a_i^r) = \epsilon \alpha a_j^{r-1} + B$, where $\epsilon = \pm 1$, $\alpha \in \rho_1$ and a_j^{r-1} does not occur in B . We construct a new system ρ' deleting a_i^r and a_j^{r-1} . Let $k: \rho \rightarrow \rho'$ be the identity on bases except $k(a_i^r) = 0$ and $k(a_j^{r-1}) = -\epsilon \alpha^{-1} B$. The boundary d' is defined by $d' = k \circ d$. When $r = 2$ we require that $d_2(a_i^2) = x(a_j^1)y$ where a_j^1 does not occur in x or y , and define $k_1(a_j^1) = (x^{-1}y^{-1})^\epsilon$. In all cases $k: \rho \rightarrow \rho'$ is a chain homotopy equivalence. $F: \rho \rightarrow \rho'$ chain homotopic to k is called an *elementary contraction of dimension r* . If ρ, ρ' are as above, a chain homotopy inverse of k is called an *elementary expansion of dimension r* .

A third elementary equivalence was introduced by Wall[11]. Let a_i^r be a basis element; we construct ρ' by changing the boundary of a_i^r . If $r \geq 3$, let $d_i^r(a_i^r) = d_r(a_i^r) + \sum_{j \neq i} \gamma_j d_r(a_j^r)$ where $\gamma_j \in Z[\bar{\rho}_1]$. If $r = 2$, let $d_i^2(a_i^2) = d_2(a_i^2) \prod_{j \neq i} x_j d_2(a_j^2) x_j^{-1}$, $x_j \in \rho_1$.

Define $k: \rho \rightarrow \rho'$ to be the identity on bases except $k(a_i^r) = a_i^r - \sum_{j \neq i} \gamma_j a_j^r$, $r \geq 3$, or if $r = 2$, $k(a_i^2) = a_i^2 - \sum_{j \neq i} x_j a_j^2$. $k: \rho \rightarrow \rho'$ is the composition of an elementary expansion and an elementary contraction, both of dimension $r + 1$. In a manifold k can be realized by isotopy of the attaching map χ_i^r so we can assign this third elementary equivalence dimension r .

A homomorphism $F: \rho \rightarrow \rho'$ is a *simple equivalence* if it is the composition of a finite sequence of elementary equivalences and simple isomorphisms. The dimension of F is the minimum over all such sequences of the maximum dimension of an elementary equivalence. The next theorem shows simple equivalence of homotopy systems reduce to the familiar theory for chain complexes.

THEOREM (Whitehead[10, Theorem 15]). *$F: \rho \rightarrow \rho'$ is a simple equivalence if and only if $F_*: C_*(\rho) \rightarrow C_*(\rho')$ is a simple chain equivalence, i.e. a chain homotopy equivalence with Whitehead torsion $\tau(F_*) = 0$.*

Next we adapt a result of Wall to control the dimension of a simple equivalence.

LEMMA 4. *Suppose ρ is a homotopy system, ρ_n has rank one over $Z[\bar{\rho}_1]$, and \mathcal{H} is a handle decomposition of M with a single n -handle. Let $n = \dim M = \dim \rho \geq 4$. If $F: \rho(M, \mathcal{H}) \rightarrow \rho$ is a monic simple equivalence then there exists a simple equivalence F' chain homotopic to F such that $\dim F' \leq n - 1$.*

Proof. Wall proved that a simple homotopy equivalence of finite connected cell complexes $\phi: K_0 \rightarrow K_1$ is homotopic to a formal deformation $D: K_0 \rightarrow K_1$ with $\dim D \leq \max \{\dim K_0, \dim K_1, 3\}$ [11]. We adapt this result to our algebraic setting. Collapse the transverse discs of \mathcal{H} to obtain a cell complex H such that $\rho(H) \approx \rho(M, \mathcal{H})$. By a result of Whitehead[10, Theorem 17] there exists a finite cell complex X , $\rho(X) \approx \rho$, and a cellular simple homotopy equivalence $F: H \rightarrow X$ such that $f_* = F$. Therefore there exists a formal deformation $D: H \rightarrow X$, $\dim D \leq n$, and the elementary deformations in D induce elementary equivalences of the same dimensions.

If F is monic by a change of basis $F(a^n) = (a^n)'$. Truncate both homotopy systems at dimension $(n - 1)$ and consider $F/n - 1: \rho(M, \mathcal{H})/n - 1 \rightarrow \rho/n - 1$. $\tau((F/n - 1)_*) = 0$ so $F/n - 1$ is also a simple equivalence[5, #3.16]. Therefore, $F/n - 1$ is chain homotopic to a simple equivalence G of dimension $\leq n - 1$. Let $G = (G^k \circ \dots \circ G^1)$, $G^i: \rho^{i-1} \rightarrow \rho^i$, a composition of elementary equivalences and simple isomorphisms. We construct $\hat{G}^i: \hat{\rho}^{i-1} \rightarrow \hat{\rho}^i$ extending each G^i to dimension n . Since $(F/n - 1) \approx G$, $F_{n-1}(d_n(a^n)) = G_{n-1}(d_n(a^n))$. Let $\hat{\rho}_n^i$ be generated by a^n , and let $\hat{G}_n^i(a^n) = a^n$. $\hat{d}_n^i(a^n) = G_{n-1}^i(\hat{d}_n^{i-1}(a^n))$. \hat{G}^i is an elementary equivalence with unchanged dimension. If $F' = \hat{G}^k \circ \dots \circ \hat{G}^1$ then $F' \approx F$ and $\dim F' \leq n - 1$. \square

If $F: \rho(M, \mathcal{H}) \rightarrow \rho'$ is an elementary equivalence and $\dim F \leq n - 3$, F can be realized in M directly. When $\dim F \geq n - 2$, it is necessary to use the homotopy system of the dual decomposition \mathcal{H}' .

LEMMA 5. *Let C_*, D_* be based $Z[\Pi_1]$ complexes of dimension n , rank $C_n = \text{rank } D_n = 1$, and $G: C_* \rightarrow D_*$ a monic chain map. If G is a simple equivalence so is $G^*: D^* \rightarrow C^*$.*

Proof. Recall that G is a simple chain equivalence iff there exist collapsible complexes B_* , B'_* , and a simple isomorphism $F: C_* \oplus B_* \rightarrow D_* \oplus B'_*$ such that the following diagram commutes up to chain homotopy:

$$\begin{array}{ccc} C_* \oplus B_* & \xrightarrow{F} & D_* \oplus B'_* \\ \uparrow & & \downarrow \\ C_* & \xrightarrow{G} & D_* \end{array}$$

Apply $\text{Hom}_{Z[\Pi_1]}(-, Z[\Pi_1])$ to this diagram. Clearly $(C_* \oplus B_*)^* \cong C^* \oplus B^*$, $(D_* \oplus B'_*)^* \cong D^* \oplus (B')^*$, and B^* , $(B')^*$ are collapsible. It is not hard to show that F^* is a simple isomorphism and $(\text{inc})^*$ is a retraction [5, 3.15]. Since $(B')^*$ is collapsible $(k)^*$ is chain homotopic to the inclusion. Therefore G^* is a simple chain equivalence. \square

Therefore if $F: \rho(M, \mathcal{H}) \rightarrow \rho$ is an elementary equivalence, $\dim F \geq n - 2$, $F^*: C^*(\rho) \rightarrow C^*(\tilde{M}, \tilde{\mathcal{H}})$ is a simple chain equivalence. F^* may cancel a pair of dual handles, a_i^r, a_i^{r-1} , $r = 2$ or 3 . In order to use our cancelling lemma we need to lift F^* to homotopy systems.

$$\begin{array}{ccc} C^*(\rho) & \xrightarrow{F^*} & C^*(\tilde{M}, \tilde{\mathcal{H}}) \approx C_*(\tilde{M}, \tilde{\mathcal{H}}') \\ & & \uparrow \\ & & \rho(M, \mathcal{H}'). \end{array}$$

We need to know that there exists a homotopy system ρ' such that $C^*(\rho) \approx C_*(\rho')$.

Definition. A based $Z[\Pi_1]$ chain complex C_* admits homotopy groups if there exists a homotopy system ρ such that $C_*(\rho) \approx C_*$.

I do not know when this is true in general but the following lemma is sufficient for our purpose.

LEMMA 6. Let C be a based $Z[\Pi_1]$ complex such that $\text{rank } C_0 = 1$, $H_0(C) \cong Z$, $H_1(C) = 0$. Let $\{a_1^1, \dots, a_k^1\}$ be a preferred basis of C_1 and suppose $\partial_1(a_i^1) = (x_i - 1)a^0$ where $x_i \in \Pi_1$ and a^0 is the preferred basis element of C_0 . Then there exists a collapsible complex B_* such that $C_* \oplus B_*$ admits homotopy groups.

Proof. $C_0 \cong Z[\Pi_1]$ so the bottom of the complex looks like a free resolution of Z as a $Z[\Pi_1]$ -module

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} Z[\Pi_1] \longrightarrow Z \longrightarrow 0.$$

The question is whether this resolution arises from a free presentation of Π_1 . Let I be the augmentation ideal $I[\Pi] \subset Z[\Pi_1]$. By assumption $\partial_1(C_1) \subset Ia^0$ and $H_0(C_*) \cong Z$ so $\partial_1(C_1) = Ia^0$ and x_1, \dots, x_k generate Π_1 . Let F be the free group on formal generators $[a_i^1], \dots, [a_k^1]$ and let $q: F \rightarrow \Pi_1$ be the homomorphism $[a_i^1] \rightarrow x_i$. Let $h_1: F \rightarrow C_1$ be the crossed homomorphism associated with q determined by $[a_i^1] \rightarrow a_i^1$. Let $R = \text{kernel } q$,

and $K = \text{kernel } \partial_1 = \partial_2(C_2)$

$$\begin{array}{ccccccc} 1 & \longrightarrow & R & \longrightarrow & F & \xrightarrow{q} & \Pi_1 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0. \end{array}$$

Since Π_1 is a finitely generated group R is finitely generated as a normal subgroup of F . Let $R = \langle r_1, \dots, r_h \rangle^F$. Build a geometric complex X as follows: let X_1 be a wedge of k circles and let X_2 be built by attaching 2-cells according to the words r_1, \dots, r_h in 1-cells. Then $\Pi_1(X_1, X_0) \cong F$ and X realizes q , h_1 , and ∂_1 . Therefore it follows from [2, Theorem 8] that h_1/R is onto K and kernel $h_1 = [R, R]$.

Let $\{a_1^2, \dots, a_m^2\}$ be a preferred basis of C_2 ; so $\{\partial_2(a_1^2), \dots, \partial_2(a_m^2)\}$ generate K as a submodule of C_1 . Choose elements $v_1, \dots, v_m \in R$ such that $h_1(v_i) = \partial_2(a_i^2)$, $i = 1, \dots, m$, and let $N = \langle v_1, \dots, v_m \rangle^F$. Then $N \subset R$ and $h_1(N) = K = h_1(R)$. If $N = R$, C_* admits homotopy groups but I do not see how to prove this. Otherwise there exist $w_i \in N$ such that $h_1(w_i) = h_1(r_i)$, $i = 1, \dots, h$, and $r_i w_i^{-1} = s_i \in [R, R]$. Then $R = \langle v_1, \dots, v_m, s_1, \dots, s_h \rangle^F$.

Let B_* be the collapsible $Z[\Pi_1]$ -complex $0 \rightarrow B_3 \rightarrow B_2 \rightarrow 0$ with preferred bases $\{b_1^3, \dots, b_h^3\}, \{b_1^2, \dots, b_h^2\}$, where $\partial_3(b_i^3) = b_i^2$. Then $C_* \oplus B_*$ admits homotopy groups. Let ρ_1 be F above and let ρ_2 have formal generators $\{[a_1^2], \dots, [a_m^2], [b_1^2], \dots, [b_h^2]\}$. Define $d_2[a_i^2] = v_i$, $i = 1, \dots, m$, and $d_2[b_i^2] = s_i$, $i = 1, \dots, h$. Therefore $d_2[\rho_2] = R$, and there exist elements $A_i \in [\rho_2, \rho_2]$, $i = 1, \dots, h$, such that $d_2(A_i) = s_i$. If $\{a_1^3, \dots, a_k^3\}$ is a preferred basis of C_3 let ρ_3 have formal generators $\{[a_1^3], \dots, [a_k^3], [b_1^3], \dots, [b_h^3]\}$. Let $d_3[b_i^3] = [b_i^2] - A_i$, $i = 1, \dots, h$, so $d_2 d_3 = 1$. $d_3[a_i^3]$ must satisfy $h_2(d_3[a_i^3]) = \partial_3(a_i^3)$ and $d_2 d_3[a_i^3] = 1$. By [2, Lemma 5] there exists a unique element of ρ_2 which satisfies this.

It is easy to check that $\rho_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1$ are the required homotopy groups. \square

§4. THE CHAIN COMPLEXES OF M

Let \mathcal{H} be a given handle decomposition, with handles ϕ_i^k attached by embeddings $\chi_i^k: (S_i^{k-1} \times D_i^{n-k}) \rightarrow \partial M_{k-1}$. By a small perturbation we can assume all the intersections $\chi_i^k(S_i^{k-1} \times 0) \cap (0 \times S_j^{n-k})$ are transverse in ∂M_{k-1} . Recall that if $\chi, \psi: (S_i^{k-1} \times D_i^{n-k}) \rightarrow \partial M_{k-1}$ are isotopic, then $M_{k-1} \cup_{\psi} \phi_i^k$ is diffeomorphic to $M_{k-1} \cup_{\chi} \phi_i^k$. We recall a basic lemma of Smale for modifying a handle decomposition.

LEMMA 7 (Smale [12]). *Let X be a manifold with boundary and $X + \phi_j^{k-1} + \phi_i^k$ the manifold obtained by attaching handles $\phi_i^k \phi_j^{k-1}$. If $\chi_i^k(S_i^{k-1} \times 0) \cap (0 \times S_j^{n-k})$ is a single point when $X \cong X + \phi_j^{k-1} + \phi_i^k$.*

Let $\partial_k: C_k(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C_{k-1}(\tilde{M}, \tilde{\mathcal{H}})$ be given by a matrix (∂_{ij}^k) . It follows easily from the proof of Lemma 1 that, provided $\dim M \geq 6$ and $4 \leq k \leq n-3$, if $\partial_{ij}^k = \pm \alpha$, $\alpha \in \Pi_1$, then χ_i^k is isotopic to an embedding such that $\chi_i^k(S_i^{k-1} \times 0) \cap (0 \times S_j^{n-k})$ is a single point. When $k = 3, 2$, we use the boundary of the homotopy system $\rho(M, \mathcal{H})$. When $k = 2$ the proof of Lemma 2 adapts to show that if $d_3[\phi_i^3]$ is represented by a word $(W \pm x[\phi_j^2] + V)$, where $x \in \rho_1$ and $[\phi_j^2]$ does not appear in the words W, V , then χ_i^3 is isotopic to an embedding such that $\chi_i^3(S_i^2 \times 0) \cap (0 \times S_j^{n-3})$ is a single point. Provided \mathcal{H} has a single 0-handle we obtain a corresponding result for 1-handles using $d_2: \rho_2(M, \mathcal{H}) \rightarrow \rho_1(M, \mathcal{H})$.

LEMMA 8. Suppose $\dim M \geq 6$, \mathcal{H} a handle decomposition of M with one 0-handle, ρ a homotopy system and $F: \rho(M, \mathcal{H}) \rightarrow \rho$ an elementary equivalence. If $\dim F \leq n-3$ there exists a handle decomposition \mathcal{H} of M such that $\rho(M, \mathcal{H}) \approx \rho$. If \mathcal{H} has one n -handle and F is monic there exists a diffeomorphism g isotopic to Id_M such that $g_* = F$.

Proof. Suppose first F is a contraction killing a pair $\phi_i', \phi_j'^{-1}$ where $d_r[\phi_i'] = \epsilon\alpha[\phi_j'^{-1}] + B$, and $k: \rho(M, \mathcal{H}) \rightarrow \rho$ is the natural retraction. By Lemmas 1 and 2 χ_i' can be isotoped so $\chi_i'(S_i'^{-1} \times 0) \cap (0 \times S_j'^{n-r})$ is a point, and by Lemma 7 the pair cancel. Think of \mathcal{H} as arising from a Morse function with a gradient like vector field η , and suppose $\phi_i', \phi_j'^{-1}$ correspond to critical points p, q . Since $\chi_i'(S_i'^{-1} \times 0) \cap (0 \times S_j'^{n-r})$ is a point, there is exactly one trajectory T of η from p to q . We alter η in a small neighborhood $N(T)$ to obtain a gradient like vector field η' which is never zero on $N(T)$ [4, Theorem 5.4]. It is not hard to see that the handle decomposition $\mathcal{H} = \{N_0 \subset \dots \subset N_n\}$ associated with η' realizes ρ . Deform Id_M along the flow lines of η' to obtain a diffeomorphism h such that $h(M_i) \subset \text{int } N_i$. During the isotopy the image of $(D_i' \times 0)$ flows down into the $(r-1)$ skeleton of \mathcal{H} so $h_\#[\phi_i'] = 0$, while the image of $(D_j'^{-1} \times 0)$ flows across what was $(D_i' \times 0)$ so $h_\#[\phi_j'^{-1}] = -\epsilon\alpha^{-1}B$. Therefore $h_* = k$, and by assumption $F \approx k$. If F is monic by Lemma 3 there exists a diffeomorphism g isotopic to h such that $g_* = F$.

If F is an elementary expansion we run this argument in reverse. If it is an elementary equivalence of the third kind the conclusion follows from the two previous cases. (In case $\dim F = n-3$ and F alters the boundary of $\phi_i'^{n-3}$, when we introduce the $(n-2)$ handle $\phi_i'^{n-2}$ we can arrange that $\chi_i'^{n-2}(S_i'^{n-3} \times 0) \cap (0 \times S_j'^2)$ is a single point). \square

THEOREM 2. Suppose $\dim M \geq 6$ and \mathcal{H} is a handle decomposition of M with one 0- and one n -handle. Let C_* be a based $Z[\Pi_1]$ complex such that both C_* and C^* admit homotopy groups. C_* is realized by a handle decomposition \mathcal{H} of M if and only if there exists a monic simple chain equivalence $G: C_*(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C_*$. If ρ, ρ' are homotopy systems inducing C_*, C^* , then \mathcal{H} can be chosen to realize ρ, ρ' . If F, F' are lifts of G, G^* , then there exists a diffeomorphism $g: M \rightarrow M$ isotopic to the identity such that $g_* = F$ and $(g^{-1})_* = F'$.

Proof. Choose homotopy systems ρ, ρ' such that $C_* \approx C_*(\rho)$ and $C^* \approx C_*(\rho')$, and lift G to a simple equivalence $F: \rho(M, \mathcal{H}) \rightarrow \rho$. By Lemma 4 there exists a sequence of elementary equivalences F^1, \dots, F^k such that $F \approx (F^k \circ \dots \circ F^1)$ and for each i , F^i is monic and $\dim F^i \leq n-1$.

$$\begin{array}{ccccc} \rho(M, \mathcal{H}) & \xrightarrow{F^1} & \rho^1 & \longrightarrow \dots \longrightarrow & \rho \\ \downarrow & & & & \downarrow \\ C_*(\tilde{M}, \tilde{\mathcal{H}}) & \xrightarrow{G} & & & C_* \end{array}$$

By Lemma 8, F^i can be realized provided $\dim F^i \leq n-3$. Otherwise $(F^i)^*: C^*(\rho^i) \rightarrow C^*(\tilde{M}, \tilde{\mathcal{H}}^{i-1}) \approx C_*(\tilde{M}, (\tilde{\mathcal{H}}^{i-1})')$ is a simple chain equivalence. $C^*(\rho^i)$ satisfies the assumptions of Lemma 7 so there exists a collapsible complex B_*^i and a homotopy system $(\rho^i)'$ such that $C^*(\rho^i) \oplus B_*^i \approx C_*(\rho^i)$. Let $0: B_*^i \rightarrow C_*(\tilde{M}, (\tilde{\mathcal{H}}^{i-1})')$ be the zero chain map. Since B_*^i is collapsible $(F^i)^* \oplus 0: C^*(\rho^i) \oplus B_*^i \rightarrow C_*(\tilde{M}, (\tilde{\mathcal{H}}^{i-1})')$ is a simple chain equivalence which we can lift to a simple equivalence $(F^i)': (\rho^i)' \rightarrow \rho(M, (\mathcal{H}^{i-1})')$.

$$\begin{array}{ccccc}
 C_*(\tilde{M}, (\mathcal{H}^{j-1})') & \xleftarrow{(F^j)^*} & C^*(\rho^j) & \xleftarrow{1 \oplus 0} & C^*(\rho^j) \oplus B_*^j \\
 \uparrow & & & & \uparrow \\
 \rho(M, (\mathcal{H}^{j-1})') & \xleftarrow{(F^j)'} & (\rho^j)' & &
 \end{array}$$

Since $\dim F^j \geq n-2$ we can assume $(F^j)_i$ is a simple isomorphism for $i \leq n-4$. Therefore $(F^j)'_i$ is a simple isomorphism for $i \geq 4$. By the proof of Lemma 4, $(F^j)'$ is a composition of elementary equivalences of dimension ≤ 3 , all of which can be realized by Lemma 8.

Continuing, we may need to add a collapsible complex B_*^j each time $\dim F^j \geq n-2$. Since B_*^j lives in dimensions 2, 3 its dual $(B^j)^*$ lives in $(n-2)$, $(n-3)$, and $C_*(\rho^j) \oplus (B^j)^*$ admits homotopy groups (essentially $\rho^j \oplus (B^j)^*$).

Let B_* be the direct sum of the B_*^j . Eventually we realize $C^* \oplus B_*$ with associated homotopy system $(\rho^k)'$.

$$\begin{array}{ccc}
 C^* \oplus B_* & \longrightarrow & C^* \\
 \uparrow & & \uparrow \\
 (\rho^k)' & \longrightarrow & \rho'.
 \end{array}$$

We know in advance C^* admits homotopy groups ρ' , and $1 \oplus 0: C^* \oplus B_* \rightarrow C^*$ is a simple chain equivalence. Therefore we can kill off B_* by a sequence of elementary equivalences of dimension ≤ 3 . This realizes ρ and ρ' .

Observe that when $\dim F^j \leq n-3$ by Lemma 8 we obtain a diffeomorphism g^j isotopic to g^{j-1} , g^1 isotopic to Id_M , such that $(g^j)_* = F^j$. When we use duality we obtain $(g^j)'$ such that $(g^j)^{-1} = (F^j)'$. If $\hat{g} = g^k \circ \dots \circ g^1$ then $\hat{g}_* = G$. If F, F' are any lifts of G, G^* then by Lemma 3 there exists a diffeomorphism g isotopic to \hat{g} such that $g_* = F$ and $(g^{-1})_* = F'$.

Conversely, if $\mathcal{H} = \{M_0 \subset \dots \subset M_n\}$ and $\mathcal{H}' = \{N_0 \subset \dots \subset N_n\}$ are handle decompositions of M deform the identity to obtain a diffeomorphism g such that $g(M_i) \subset \text{int } N_i$. By Chapman's theorem [13] $g_*: C_*(M, \mathcal{H}) \rightarrow C_*(M, \mathcal{H}')$ is a simple chain equivalence, which completes the proof. \square

We will call a triple (C_*, ρ, ρ') as in Theorem 2 a *homotopy complex of M* . Given $f \in T_{\mathcal{H}}$ we will say an endomorphism (E, E') of such a complex is an *endomorphism of f* if there exists a simple chain equivalence G as in Theorem 2, and the following diagram commutes up to chain homotopy.

$$\begin{array}{ccc}
 C_*(\tilde{M}, \mathcal{H}) & \xrightarrow{G} & C_* \\
 \downarrow f_* & & \downarrow E_* \\
 C_*(\tilde{M}, \mathcal{H}') & \xrightarrow{G} & C_*.
 \end{array}$$

Returning to the original problem of minimizing entropy for fitted diffeomorphisms we obtain

THEOREM 3. *Let M, \mathcal{H} be as in Theorem 2, and $f \in T_{\mathcal{H}}$. If (E, E') is an endomorphism of f then f is isotopic to a fitted diffeomorphism f' such that*

$$h(f') = \max \{ \log s(|E|), \log s(|E'|) \}.$$

Proof. Suppose (E, E') is defined on (C_*, ρ, ρ') , and $G: C_*(\tilde{M}, \tilde{\mathcal{H}}) \rightarrow C_*$ is a simple chain equivalence as above. Let $\mathcal{H} = \{N_0 \subset \cdots \subset N_n\}$ realize (C_*, ρ, ρ') and let $H: C_* \rightarrow C_*(\tilde{M}, \tilde{\mathcal{H}})$ be a chain homotopy inverse of G . By Theorem 2 there exist diffeomorphisms g, h isotopic to Id_M such that $g_* = G$ and $h_* = H$. f is isotopic to $g \circ f \circ h$ and by assumption $(g \circ f \circ h)_* \simeq E_*$. By Lemma 3 f is isotopic to $k \in T_{\mathcal{H}}$ such that $k_* = E$ and $(k^{-1})_* = E'$. The result now follows from Theorem 1. \square

This theorem identifies the infimum of the entropy of fitted diffeomorphisms, with a single source and a single sink, in the component of f :

$$\text{infimum} = \inf_{(E, E')} \max \{ \log s(|E|), \log s(|E'|) \}$$

where (E, E') range over endomorphisms of f . If $f \in \text{Diff}^r(M)$ is arbitrary, we pick a handle decomposition \mathcal{H} of M and deform f in any way to $f_1 \in T_{\mathcal{H}}$. The minimum entropy detected is independent of the choice of \mathcal{H} and of f_1 .

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